Hence for a similarity solution either $D' = 0$ or $D'(x)/D(x) = \alpha D^2, \alpha \neq 0$. In the former case $\eta = \text{const}\, y$. Equation (4) then gives $F''(\eta) = 0$, and its solution $F = A + B\eta$ clearly fails to satisfy $F(\infty) = F'(0) = 0$ unless $A = B = 0$, in which case, $F(\eta) \equiv 0$. In the latter case $D(x) \sim x^{-1/2}$ so that $\eta \sim yx^{-1/2}$. Now taking $\eta = yU^{1/2}/2(\nu x)^{1/2}$, Eq. (2) becomes

$$F'' + 2\eta F' = 0$$

which on integration gives $F'(\eta) = A_1 e^{-\eta^2}$. Since $F'(0) = 0$, $F'(\eta) \equiv 0$, and its solution satisfying $F(\infty) = 0$ is clearly $F(\eta) \equiv 0$. Thus there is no nontrivial similarity solution of Eq. (2) satisfying $F(\infty) = F'(0) = 0$.

The same conclusion can be reached from Eq. (6) of my note. When $UCC'/\nu = 0$, Eq. (6) can be put in the form

$$f'' + \lambda_1 f = 0 \qquad (5)$$

λ_1 being a constant, and Eq. (5) clearly has no nontrivial solution satisfying $f(\infty) = 0$; $f'(0) = 0$ for all the three cases $\lambda_1 > 0$, $\lambda_1 = 0$, and $\lambda_1 < 0$.

Comment on "Basis for Derivation of Matrices for the Direct Stiffness Method"

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AND

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In a recent article, Melosh[1] presents a stiffness matrix for a rectangular plate element that may be used for plate bending analysis. There are several errors in the stiffness matrix that should be corrected. Also the matrix can be presented in a more compact form by noting that the stiffness coefficients for nodes 2, 3, and 4 of the plate can be obtained by proper permutations and sign changes of the stiffness coefficients for node 1. These relations are given by the stiffness matrix in Fig. 1. The coefficients describing the behavior of node 1 (the first three columns of the stiffness matrix) are given in Fig. 2. It should be noted that for an isotropic plate, the coefficients in the moment-curvature relationship have the following values:

$$A_{11} = A_{22} = D$$
$$A_{21} = A_{12} = \nu D$$
$$A_{33} = (1 - \nu)D$$

$$\begin{pmatrix} Z_1 \\ M_{x1}/a \\ M_{y1}/b \\ Z_2 \\ M_{x2}/a \\ M_{y2}/b \\ Z_3 \\ M_{x3}/a \\ M_{y3}/b \\ Z_4 \\ M_{x4}/a \\ M_{y4}/b \end{pmatrix} = \begin{pmatrix} K_{11} & & & & & & \text{SYMMETRIC} \\ K_{21} & K_{22} \\ K_{31} & K_{32} & K_{33} \\ K_{41} & K_{42} & K_{43} & K_{11} \\ K_{51} & K_{52} & K_{53} & K_{21} & K_{22} \\ K_{61} & K_{62} & K_{63} & -K_{31} & -K_{32} & K_{33} \\ K_{71} & K_{72} & K_{73} & K_{10,1} & K_{10,2} & -K_{10,3} & K_{11} \\ K_{81} & K_{82} & K_{83} & K_{11,1} & K_{11,2} & -K_{11,3} & -K_{21} & K_{22} \\ K_{91} & K_{92} & K_{93} & -K_{12,1} & -K_{12,2} & K_{12,3} & -K_{31} & K_{32} & K_{33} \\ K_{10,1} & K_{10,2} & K_{10,3} & K_{71} & K_{72} & -K_{73} & K_{41} & -K_{42} & -K_{43} & K_{11} \\ K_{11,1} & K_{11,2} & K_{11,3} & K_{81} & K_{82} & -K_{83} & -K_{51} & K_{52} & K_{53} & -K_{21} & K_{22} \\ K_{12,1} & K_{12,2} & K_{12,3} & -K_{91} & -K_{92} & K_{93} & -K_{61} & K_{62} & K_{63} & K_{31} & -K_{32} & K_{33} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ a w_{1,x} \\ b w_{1,y} \\ w_2 \\ a w_{2,x} \\ b w_{2,y} \\ w_3 \\ a w_{3,x} \\ b w_{3,y} \\ w_4 \\ a w_{4,x} \\ b w_{4,y} \end{pmatrix}$$

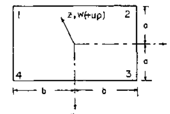

Fig. 1 Basic stiffness coefficient relationships.

$$\begin{pmatrix} K_{11} \\ K_{21}\ K_{22} \\ K_{31}\ K_{32}\ K_{33} \\ K_{41}\ K_{42}\ K_{43} \\ K_{51}\ K_{52}\ K_{53} \\ K_{61}\ K_{62}\ K_{63} \\ K_{71}\ K_{72}\ K_{73} \\ K_{81}\ K_{82}\ K_{83} \\ K_{91}\ K_{92}\ K_{93} \\ K_{10,1}\ K_{10,2}\ K_{10,3} \\ K_{11,1}\ K_{11,2}\ K_{11,3} \\ K_{12,1}\ K_{12,2}\ K_{12,3} \end{pmatrix} = \left(\frac{b}{6a^3}\right) A_{11} \begin{pmatrix} 6 \\ 6 & 8 \\ 0 & 0 & 0 \\ 3 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \\ -3 & -3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 0 \\ -6 & -6 & 0 \\ 6 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \left(\frac{a}{6b^3}\right) A_{22} \begin{pmatrix} 6 \\ 0 & 0 \\ 6 & 0 & 8 \\ -6 & 0 & -6 \\ 0 & 0 & 0 \\ 6 & 0 & 4 \\ -3 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 2 \\ 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 4 \end{pmatrix} + \left(\frac{1}{2ab}\right) A_{21} \begin{pmatrix} 1 \\ 1 & 0 \\ 1 & 2 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \left(\frac{1}{30ab}\right) A_{33} \begin{pmatrix} 21 \\ 3 & 8 \\ 3 & 0 & 8 \\ -21 & -3 & -3 \\ -3 & -8 & 0 \\ 3 & 0 & -2 \\ 21 & 3 & 3 \\ -3 & 2 & 0 \\ -3 & 0 & 2 \\ -21 & -3 & -3 \\ 3 & -2 & 0 \\ -3 & 0 & -8 \end{pmatrix}$$

Fig. 2 Stiffness coefficients for node 1.

where D represents the flexural rigidity of the plate,

$$D = Et^3/12(1 - \nu^2)$$

Numerical convergence studies of this stiffness matrix have indicated that it performs adequately, although the monotonic behavior predicted by Melosh does not always occur. Table 1 gives numerical factors for the maximum deflection of a square plate that illustrate convergence behavior of this element. The clamped plate with a central concentrated load does not behave monotonically for a very coarse "mesh." This lack of monotonicity is probably because the four-element approximation of the clamped plate cannot differentiate between a central concentrated load and a uniformly distributed load. In both cases there is only one equation in one unknown to be solved.

It also should be noted that convergence may occur from "above" as well as from "below." This is due to lack of slope continuity along adjacent edges of the plate elements, as was pointed out by Pian.[2] If the element edges were continuous with respect to displacements and slopes (not just displacements), then the elements would always be stiffer than the actual plate and convergence would always be from "below."

Table 1 Computed central deflection of a square plate for several "meshes"[a]

Mesh size	Total no. of nodes	Simply supported plate		Clamped plate	
		α (Uniform load)	β (Concentrated load)	α (Uniform load)	β (Concentrated load)
(2×2)	9	0.003446	0.013784	0.001480	0.005919
(4×4)	25	0.003939	0.012327	0.001403	0.006134
(8×8)	81	0.004033	0.011829	0.001304	0.005803
(12×12)	169	0.004050	0.011715	0.001283	0.005710
(16×16)	289	0.004056	0.011671	0.001275	0.005672
Exact (Timoshenko)		0.004062	0.01160	0.00126	0.00560

[a] $w_{\max} = \alpha q a^4/D$ for a uniformly distributed load q; $w_{\max} = \beta P a^2/D$ for a central concentrated load P.

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[1] Melosh, R. J., "Basis for derivation of matrices for the direct stiffness method," AIAA J. **1**, 1631–1637 (1963).

[2] Pian, T. H. H., "Derivation of element stiffness matrices by assumed stress distributions," AIAA J. **2**, 1333–1336 (1964).

Reply by Author to J. L. Tocher and K. K. Kapur

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TOCHER and Kapur have corrected algebraic and typographical errors in the plate stiffness matrix based on the function given in Ref. 1. They have demonstrated that this matrix has no advantages over that given in Ref. 2.

Shortcomings of this matrix are due to the error in deriving the Lagrange-Hermite first-order interpolation formula.[3] Schmidt[4] has obtained the correct form of this formula and derived the corresponding stiffness matrix. The correct form does satisfy the continuity requirements, includes rigid body modes, and provides monotonic convergence.

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